

n/p-harmonic maps: regularity for the sphere case

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We introduce n/p -harmonic maps as critical points of the energy

$$\mathcal{E}_{n,p}(v) = \int_{\mathbb{R}^n} |\Delta^{\frac{\alpha}{2}} v|^p$$

where pointwise $v : D \subset \mathbb{R}^n \rightarrow \mathbb{S}^{N-1}$, for the N -sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ and $\alpha = \frac{n}{p}$. This energy combines the non-local behaviour of the fractional harmonic maps introduced by Riviere and the first author with the degenerate arguments of the n -laplacian. In this setting, we will prove Hölder continuity.

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1 Introduction

Our work is motivated by recent results [DLR11b], [DLR11a], [Sch12], [DL10], [Sch11] which proved regularity for critical points of the energy \mathcal{F}_n acting on maps $v : \mathbb{R}^n \rightarrow \mathbb{R}^N$,

$$\mathcal{F}_n(v) = \int_{\mathbb{R}^n} |\Delta^{\frac{n}{4}} v|^2 \quad v \in \mathcal{N} \subset \mathbb{R}^N \text{ a.e.}$$

Here, the operator $\Delta^{\frac{\alpha}{2}} v$ is defined as a multiplier operator with symbol $-|\xi|^\alpha$, that is, denoting the Fourier transform and its inverse by $()^\wedge$ and $()^\vee$, respectively,

$$\Delta^{\frac{\alpha}{2}} v = (-|\xi|^\alpha v^\wedge)^\vee.$$

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These energies were introduced by T. Rivière and the first author – and they can be seen as an n -dimensional alternative to the two-dimensional Dirichlet energy

$$\mathcal{D}_2(v) = \int_{\mathbb{R}^2} |\nabla v|^2 \quad v \in \mathcal{N} \subset \mathbb{R}^N \text{ a.e.}$$

Both energies have critical Euler-Lagrange equations. That is, the highest order terms scale exactly as the lower-order terms, thus inhibiting the application of a general regularity theory based only on the general growth of the right-hand side – one has to consider the finer behavior of the equation: These exhibit an antisymmetric structure, which is closely related to the appearance of Hardy spaces and compensated compactness – and induces regularity of critical points. In two dimensions, these facts were observed in Rivière’s celebrated [Riv07] for all conformally invariant variational functionals (of which the Dirichlet energy is a prototype). We refer the interested reader to the introductions of [DLR11b], [DLR11a] for more on this.

Another possibility of generalizing the Dirichlet energy to arbitrary dimensions (whilst preserving the criticality of the Euler-Lagrange equations) is to consider

$$\mathcal{D}_n(v) = \int_{\mathbb{R}^n} |\nabla v|^n \quad v \in \mathcal{N} \subset \mathbb{R}^N \text{ a.e.}$$

Again in this case, the now degenerate Euler-Lagrange equations are critical and exhibit an antisymmetric structure, cf. [Riv08, Chapter III] – only that it is not known so far, whether in general this structure implies even continuity. In fact, towards regularity of these systems, only few results are known. In [Str94] P. Strzelecki proved regularity, if the target manifold is a round sphere \mathbb{S}^{N-1} – which extended the respective Dirichlet-energy result by F. Hélein [Hél90]. In the setting of general manifolds, we know so far of convergence results, cf. [Wan05], and only under additional assumptions on the solution there are regularity results, cf. [DM10], [Kol10], [Sch10a].

It then seems interesting to consider an energy which combines the difficulties of \mathcal{D}_n and \mathcal{F}_n . Namely we will work with

$$\mathcal{E}_{n,\bar{p}}(v) = \int_{\mathbb{R}^n} \left| \Delta^{\frac{\bar{\alpha}}{2}} v \right|^{\bar{p}} \quad v|_D \in \mathcal{N} \subset \mathbb{R}^N \text{ a.e., where } \bar{\alpha} = \frac{n}{\bar{p}}, D \subset \subset \mathbb{R}^n \quad (1.1)$$

Note, in the Euler-Lagrange equations of $\mathcal{E}_{n,\bar{p}}$, the leading order differential operator is nonlocal *and* degenerate. Again, these settings are critical for regularity: One checks that any mapping v with finite energy $\mathcal{E}_{n,\bar{p}}(v) < \infty$ belongs to BMO, but does not necessarily need to be continuous, as Frehse’s counterexample [Fre73] shows.

Here we consider the situation of a sphere, i.e. $\mathcal{N} = \mathbb{S}^{N-1}$. Our main result is:

Theorem 1.1. *Let u be a critical point of $\mathcal{E}_{n,\bar{p}}$ as in (1.1). Then u is Hölder-continuous.*

Naturally, one expects this result to hold at least partially for more general manifolds \mathcal{N} . To this end, in [DLS12] we will treat this case of general manifolds, but with the condition $\bar{p} \leq 2$.

The proof relies on a suitable adaption of the arguments in [DLR11b], [DLR11a], [Sch12], [DL10], [Sch11], the details of which we will explain in the next section: The Euler-Lagrange equations of a critical point, see [DLR11b], [Sch12], imply that

$$\int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \Delta^{\frac{\bar{\alpha}}{2}} (\omega_{ij} u^j \varphi) = 0 \quad \text{for all } \varphi \in C_0^\infty(D), \omega_{ij} = -\omega_{ji} \in \mathbb{R}. \quad (1.2)$$

Note, that the main difference and difficulty comparing this equation to the $n/2$ -harmonic case in [DLR11b], [Sch12], is the weight $|\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2}$! Moreover, we have the sphere-condition,

$$|u(x)| = 1 \quad \text{for a.e. } x \in D. \quad (1.3)$$

For a sketch of the proof, let us assume that $D = \mathbb{R}^n$. Note, that (1.3) reveals information about the growth of derivatives of u in the direction of u :

$$u \cdot \nabla u \equiv 0$$

Moreover – and more suitable to our case –

$$-2u \cdot \Delta^{\frac{\bar{\alpha}}{2}} u = \left(\Delta^{\frac{\bar{\alpha}}{2}} |u|^2 - u \cdot \Delta^{\frac{\bar{\alpha}}{2}} u - u \cdot \Delta^{\frac{\bar{\alpha}}{2}} u \right) - \underbrace{\Delta^{\frac{\bar{\alpha}}{2}} |u|^2}_{\equiv 0}. \quad (1.4)$$

We set

$$H_\alpha(u, v) := \Delta^{\frac{\alpha}{2}}(uv) - u\Delta^{\frac{\alpha}{2}}v - v\Delta^{\frac{\alpha}{2}}u,$$

We will see that 3-term commutators $H_\alpha(u, v)$, appearing on the right-hand side of (1.4), are more regular than each of their three generating terms. Compensation phenomena for 3-term commutators were first observed by the first author and Rivière in [DLR11b] and [DLR11a] in the context of half-harmonic maps by using the so-called para-products. Such compensation phenomena can be formulated in different ways, for instance as an expansion of lower order derivatives. This can best be seen by taking $\alpha = 2$: $H_2(u, v) = 2\nabla u \cdot \nabla v$ – they behave like products of lower-order operators applied to u and v . This interpretation has been developed by the second author in [Sch11], [Sch12] and it is the approach that we will use in this paper. We finally mention that these lower order expansions are also closely related to the T1-Theorem and the “Leibniz rule” for fractional order derivatives obtained by Kato and Ponce, see [KP88] and [Hof98, Corollary 1.2]. The necessary estimates for the operators $H_\alpha(u, v)$, can be paraphrased by

Theorem 1.2 (cf. [Sch11]). *For any $\alpha \in (0, n)$, Let $u = \Delta^{-\frac{\alpha}{2}}\Delta^{\frac{\alpha}{2}}u$, $v = \Delta^{-\frac{\alpha}{2}}\Delta^{\frac{\alpha}{2}}v$. Then for $\alpha \in (0, n)$ there exists some constant $C_\alpha > 0$ and a number $L \equiv L_\alpha \in \mathbb{N}$, and for $k \in \{1, \dots, L\}$ constants $s_k \in (0, \alpha)$, $t_k \in [0, s_k]$ such that for any $i = 1, \dots, n$, where \mathcal{R}_i denotes the Riesz transform,*

$$|\mathcal{R}_i H_\alpha(u, v)(x)| \leq C \sum_{k=1}^L M_k \Delta^{-\frac{s_k - t_k}{2}} \left(\Delta^{-\frac{t_k}{2}} |\Delta^{\frac{\alpha}{2}} u| \ N_k \Delta^{-\frac{\alpha}{2} + \frac{s_k}{2}} |\Delta^{\frac{\alpha}{2}} v| \right).$$

Here, M_k, N_k are possibly Riesz transforms, or the identity. Moreover, $|s_k - t_k|$ can be supposed to be arbitrarily small. In particular, for any $\alpha \in (0, n)$, $q, q_1, q_2 \in [1, \infty]$ such that

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|H_\alpha(u, v)\|_{(\frac{n}{\alpha}, q), \mathbb{R}^n} \prec \|\Delta^{\frac{\alpha}{2}} u\|_{(\frac{n}{\alpha}, q_2), \mathbb{R}^n} \|\Delta^{\frac{\alpha}{2}} v\|_{(\frac{n}{\alpha}, q_1), \mathbb{R}^n}. \quad (1.5)$$

Here, $\|\cdot\|_{(p, q)}$ denotes the Lorentz-space $L^{p, q}(\mathbb{R}^n)$ -norm.

Consequently, (1.3) controls $u \cdot \Delta^{\frac{\alpha}{2}} u$ roughly like

$$\|u \cdot \Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, \mathbb{R}^n} \prec \|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, \mathbb{R}^n}^2.$$

This argument can be localized, and then implies an estimate for the growth $u \cdot \Delta^{\frac{\alpha}{2}} u$ in the $L^{\bar{p}}$ -norm on small balls by the square $\|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}}^2$ localized essentially to slightly bigger balls.

Now the fact that $|u| \equiv 1$, implies also that in order to control the growth of $\Delta^{\frac{\alpha}{2}} u$, it suffices to estimate the growth of $u \cdot \Delta^{\frac{\alpha}{2}} u$ and the growth of $\omega_{ij} u^i \Delta^{\frac{\alpha}{2}} u^j$ for finitely many $\omega_{ij} = -\omega_{ji} \in \mathbb{R}$, see Proposition B.1. But terms of the form $\omega_{ij} u^i \Delta^{\frac{\alpha}{2}} u^j$ can be estimated by the Euler-Lagrange equation (1.2).

By this kind of argument, we obtain (essentially) the following growth estimates for all balls B_r

$$\|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, B_r} \leq \|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, B_{\Lambda r}}^2 + \Lambda^{-\gamma} \|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, \mathbb{R}^n} \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}},$$

for some $\gamma > 0$, and any $\Lambda > 2$. Using an iteration technique, this implies that

$$\|\Delta^{\frac{\alpha}{2}} u\|_{\bar{p}, B_r} \leq C r^\alpha,$$

which accounts for the Hölder-continuity of u .

Let us briefly underline the differences to the manifold case treated in [DLS12]. There the simple condition (1.3) does not hold anymore and we follow the approach introduced in [DLR11a] which consists in considering separately the tangential and normal projections of $\Delta^{\frac{\alpha}{2}} u$ (that is, we work with projections related to the *derivatives* of u). For the moment, this prevents us to treat the case of *extremely small* α (which in the sphere case poses no problems). On the other hand, the respective Euler-Lagrange equations actually exhibit a non-trivial right-hand side with antisymmetric structure. This will force us, to estimate the growth of $\Delta^{\frac{\alpha}{2}} u$ in the weak space $L^{\bar{p}, \infty}$,

which in turn will make it necessary to gain $L^{\bar{p},1}$ -estimates from the three-term commutators $H_{\bar{\alpha}}(\cdot, \cdot)$. This again, cf. (1.5) for $q_1 = q_2 = 2$, will only be possible if $\bar{p} \leq 2$.

We will use *notation* similar to [Sch12]:

We say that $A \subset \subset \mathbb{R}^n$ if A is a bounded subset of \mathbb{R}^n . For a set $A \subset \mathbb{R}^n$ we will denote its n -dimensional Lebesgue measure by $|A|$. By $B_r(x) \subset \mathbb{R}^n$ we denote the open ball with radius r and center $x \in \mathbb{R}^n$. If no confusion arises, we will abbreviate $B_r \equiv B_r(x)$. If $p \in [1, \infty]$ we usually will denote by p' the Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$. By $f * g$ we denote the convolution of two functions f and g . Lastly, our constants – frequently denoted by C or c – can possibly change from line to line and usually depend on the space dimensions involved, further dependencies will be denoted by a subscript, though we will make no effort to pin down the exact value of those constants. If we consider the constant factors to be irrelevant with respect to the mathematical argument, for the sake of simplicity we will omit them in the calculations, writing \prec, \succ, \approx instead of \leq, \geq and $=$.

We will use the same cutoff-functions as in, e.g., [DLR11b], [Sch12]: $\eta_r^k \in C_0^\infty(A_{r,k})$ where

$$B_{r,k}(x) := B_{2^k r}(x)$$

for $k \geq 1$,

$$A_{r,k}(x) := B_{r,k+1}(x) \setminus B_{r,k-1}(x),$$

and for $k = 0$

$$A_{r,0}(x) := B_{r,0}(x).$$

Moreover, $\sum_k \eta_r^k \equiv 1$ pointwise everywhere, and we assume that $|\nabla^l \eta_r^k| \leq C_l (2^k r)^{-l}$.

2 Proof of Theorem 1.1

Let $\bar{\alpha} \in (0, n)$, $\bar{p} = \frac{n}{\bar{\alpha}} \in (1, \infty)$, and $u \in L^{\bar{p}}(\mathbb{R}^n, \mathbb{R}^N)$, $\Delta^{\frac{\bar{\alpha}}{2}} u \in L^{\bar{p}}(\mathbb{R}^n, \mathbb{R}^N)$. Assume moreover, that $D \subset\subset \mathbb{R}^n$ such that (1.3) (1.2) holds.

As (1.2) and (1.3) are equations satisfied by any critical point u of Theorem 1.1, we have to show the following

Theorem 2.1. *Let $u \in L^{\bar{p}}(\mathbb{R}^n)$, $\Delta^{\frac{\bar{\alpha}}{2}} u \in L^{\bar{p}}(\mathbb{R}^n)$ satisfy (1.2), (1.3). Then u is Hölder continuous in D .*

In order to prove Theorem 2.1, we first rewrite equations (1.2) and (1.3) in a fashion similar to [DLR11b], [Sch12]: Firstly, Equation (1.2) is equivalent to

$$\int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} u^j \omega_{ij} \Delta^{\frac{\bar{\alpha}}{2}} u^i \Delta^{\frac{\bar{\alpha}}{2}} \varphi = - \int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \omega_{ij} H(u^j, \varphi) \quad \text{for all } \varphi \in C_0^\infty(D), \omega_{ij} = -\omega_{ji} \in \mathbb{R}. \quad (2.1)$$

Here and henceforth,

$$H(a, b) \equiv H_{\bar{\alpha}}(a, b) \equiv \Delta^{\frac{\bar{\alpha}}{2}}(ab) - a \Delta^{\frac{\bar{\alpha}}{2}} b - b \Delta^{\frac{\bar{\alpha}}{2}} a.$$

Assume we prove Hölder-continuity of u in a Ball $B \subset\subset D$. Pick a slightly bigger ball $\tilde{B} \subset\subset D$, $\tilde{B} \supset\supset B$, and let $w := \bar{\eta}u$, for some

$$\bar{\eta} \in C_0^\infty(D, [0, 1]), \quad \bar{\eta} \equiv 1 \text{ on } \tilde{B}.$$

Note that $w \in L^{\bar{p}}(\mathbb{R}^n)$ for any $p \in [1, \infty]$. It suffices to show Hölder regularity for w . The relevant equations for w stemming from (1.3) and (2.1) are then (again, cf. [DLR11b], [Sch12])

$$w \cdot \Delta^{\frac{\bar{\alpha}}{2}} w = \frac{1}{2} H(w, w) + \frac{1}{2} \Delta^{\frac{\bar{\alpha}}{2}} \bar{\eta}^2 \quad \text{a.e. in } \mathbb{R}^n, \quad (2.2)$$

and for all $\varphi \in C_0^\infty(D)$, $\omega_{ij} = -\omega_{ji} \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} w^j \omega_{ij} \Delta^{\frac{\bar{\alpha}}{2}} w^i \Delta^{\frac{\bar{\alpha}}{2}} \varphi \quad (2.3)$$

$$= \omega_{ij} \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} w^i - |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) w^j \Delta^{\frac{\bar{\alpha}}{2}} \varphi \quad (2.4)$$

$$+ \omega_{ij} \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) (w^j - u^j) \Delta^{\frac{\bar{\alpha}}{2}} \varphi \quad (2.5)$$

$$+ \omega_{ij} \int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i H(w^j - u^j, \varphi) \quad (2.6)$$

$$+ \omega_{ij} \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} w^i - |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) H(w^j, \varphi) \quad (2.7)$$

$$- \omega_{ij} \int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} w^i H(w^j, \varphi). \quad (2.8)$$

Now we need to appropriately adapt several arguments of [DLR11b], [Sch12]: First of all, using (2.2) we control $\Delta^{\frac{\bar{\alpha}}{2}} w$ projected into the orthogonal space to the sphere at the point w , $T_w^\perp \mathbb{S}^{N-1}$.

The orthogonal part

Namely, from (2.2) and Lemma A.9 one infers

Lemma 2.2. *There is $\gamma = \gamma_{\bar{\alpha}, \bar{p}} > 0$ and a constant C depending on the choice of B , \tilde{B} , $\bar{\eta}$, such that the following holds: For any $\varepsilon > 0$ there exists $\Lambda > 0$, $R > 0$, such that for any $B_{\Lambda r} \subset\subset B$, $r \in (0, R)$,*

$$\|w \cdot \Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_r} \leq \varepsilon \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{\Lambda r}} + C r^\gamma + \varepsilon \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}}.$$

The next step is to control the tangential part of $\Delta^{\frac{\bar{\alpha}}{2}} w$ by means of (2.3). The terms on the right-hand side of (2.3) can be divided into two groups. The integrands of (2.4), (2.5), (2.6), and (2.7) always contain differences of the form $w - u$ which is trivial in \tilde{B} . Consequently, we show that these terms behave subcritical.

Estimates of Tangential Part: Subcritical Terms (2.4), (2.5), (2.6), (2.7)

To be more precise, assume $|\omega_{ij}| \leq 2$. We claim that for any $\varphi \in C_0^\infty(B_r)$, where $B_{2r} \subset B$, all but the last term on the right-hand side of (2.3) can be estimated by a constant depending on u and B and the distance between B_r and $\partial\tilde{B}$ times r^γ for some γ :

Proposition 2.3 (Subcritical Terms). *There exists a constant C depending on u , B , \tilde{B} , the choice of $\bar{\eta}$, and an exponent $\gamma \equiv \gamma_{\bar{p}, \bar{\alpha}} > 0$ such that for any $\varphi \in C_0^\infty(B_r)$, $\|\Delta^{\frac{\bar{\alpha}}{2}} \varphi\|_{\bar{p}} \leq 1$ for arbitrary $B_{2r} \subset B$, if*

$$\begin{aligned} I &:= \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} w^i - |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) w^j \Delta^{\frac{\bar{\alpha}}{2}} \varphi, \\ II &:= \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) (w^j - u^j) \Delta^{\frac{\bar{\alpha}}{2}} \varphi, \\ III &:= \int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i H(w^j - u^j, \varphi), \\ IV &:= \int_{\mathbb{R}^n} \left(|\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} w^i - |\Delta^{\frac{\bar{\alpha}}{2}} u|^{\bar{p}-2} \Delta^{\frac{\bar{\alpha}}{2}} u^i \right) H(w^j, \varphi), \end{aligned}$$

then

$$|I| + |II| + |III| + |IV| \leq C r^\gamma.$$

Proof. Note that for any $x, y \in \mathbb{R}^n$ and any $a, b \in \mathbb{R}^N$ we have (cf., e.g., [Sch10b, Proposition 4.1])

$$\left| |a|^{p-2} a - |b|^{p-2} b \right| \leq C_p \begin{cases} |a - b|^{p-1} & \text{if } p \in [1, 2], \\ |a - b|^{p-1} + |a - b| |b|^{p-2} & \text{if } p > 2. \end{cases}$$

Now we argue via Proposition 2.4. In order to do so, we use that there is some positive distance $d \equiv d_{B, \tilde{B}} > 0$, such that $\text{dist}(\text{supp } \varphi, \mathbb{R}^n \setminus \tilde{B}) > d$. This is straight-forward for I and IV (recall that $w \in L^\infty$ for I and use Proposition A.7 for IV). For III , we apply Proposition 2.4 to the terms,

$$H(w^j - u^j, \varphi) = H((\bar{\eta} - 1)u^j, \varphi) = \varphi \Delta^{\frac{\bar{\alpha}}{2}}((\bar{\eta} - 1)u^j) + (\bar{\eta} - 1)u^j \Delta^{\frac{\bar{\alpha}}{2}} \varphi,$$

and finally, for II we first estimate

$$\|(1 - \bar{\eta})u \Delta^{\frac{\bar{\alpha}}{2}} \varphi\|_{\bar{p}} \prec \|u\|_{\bar{p}} \|(1 - \bar{\eta})\Delta^{\frac{\bar{\alpha}}{2}} \varphi\|_{\infty}.$$

□

In the proof of Proposition 2.3 we used the following estimate, which can be proven by an argument which appears in a similar form already in [DLR11b].

Proposition 2.4 (Estimates for disjoint-support terms). *Let $r \in (0, 1)$, $d > 0$, $p, q \in [1, \infty)$ such that $B_{r+d} \subset \tilde{B}$. Then, for any $f \in L^q(\mathbb{R}^n)$, with $\Delta^{\alpha/2} f \in L^q(\mathbb{R}^n)$,*

$$\left\| \Delta^{\frac{\alpha}{2}}((1 - \bar{\eta})f) \right\|_{p, B_r} \prec C_d r^{\frac{n}{p}} (\|f\|_{q, \mathbb{R}^n} + \|\Delta^{\frac{\alpha}{2}} f\|_{q, \mathbb{R}^n}). \quad (2.9)$$

And if moreover $\text{supp } f \subset B_r$, for some $\gamma = \gamma_\alpha$,

$$\left\| (1 - \bar{\eta})\Delta^{\frac{\alpha}{2}} f \right\|_{\infty, \mathbb{R}^n} \prec C_d r^\gamma \left\| \Delta^{\frac{\alpha}{2}} f \right\|_q. \quad (2.10)$$

Proof. Instead of the argument using Fourier-transform as in [DLR11b], [Sch12] etc., we use the following argument: Let $\alpha =: K/2 + s$, where $s \in (0, 2)$ (the case $s = 0$ is trivial), $K \in 2\mathbb{N}$, that is

$$\Delta^{\frac{\alpha}{2}} = \Delta^{\frac{s}{2}} \Delta^K.$$

Set $h := \Delta^K g$, and recall that

$$\Delta^{\frac{s}{2}} h(x) = c \int \frac{h(x+z) + h(x-z) - 2h(x)}{|z|^{n+s}} dz$$

Thus, if $\text{supp } g \subset \mathbb{R}^n \setminus B_{r+d}$ and $x \in B_r$, as is the case in (2.9), or $\text{supp } g \subset B_r$ and $x \in \mathbb{R}^n \setminus B_{r+d}$, as is the case in (2.10)

$$|\Delta^{\frac{s}{2}} h(x)| \prec \int_{|z|>d} |h(x+z)| |z|^{-n-s} dz = |h| * \left(|\cdot|^{-n-s} \chi_{|\cdot|>d} \right)(x),$$

and

$$\begin{aligned} \| |h| * \left(|\cdot|^{-n-s} \chi_{|\cdot|>d} \right) \|_p &\prec \|h\|_q d^{-n(q_2-1)-2sq_2} \\ &\prec d^{-n(q_2-1)-2sq_2} (\|f\|_{q,\mathbb{R}^n} + \|\Delta^K f\|_{q,\mathbb{R}^n}), \end{aligned}$$

where q_2 is chosen such that

$$1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{q_2}.$$

For (2.9) we then use the fact that $W^{K,q} \subset W^{\alpha,q}$, that is

$$\|f\|_{q,\mathbb{R}^n} + \|\Delta^K f\|_{q,\mathbb{R}^n} \prec \|f\|_{q,\mathbb{R}^n} + \|\Delta^{\frac{\alpha}{2}} f\|_{q,\mathbb{R}^n}.$$

For (2.9), note that by Poincaré's inequality,

$$\|f\|_{q,\mathbb{R}^n} + \|\Delta^K f\|_{q,\mathbb{R}^n} \prec \|\Delta^K f\|_{q,\mathbb{R}^n} \prec r^{\alpha-K} \|\Delta^{\frac{\alpha}{2}} f\|_{q,\mathbb{R}^n}.$$

□

Estimates of Tangential Part - The Critical Term (2.8)

In order to estimate the tangential part of $\Delta^{\frac{\alpha}{2}} w$ completely, we need to control the last term (2.8), which is done in the following two propositions.

Proposition 2.5. *There is a constant $C_{\overline{\alpha}}$, $R > 0$ and an exponent $\gamma > 0$ such that the following holds. Let $\|w\|_{\infty} \leq 1$, and assume $\Delta^{\frac{\alpha}{2}} w \in L^{\overline{p}}(\mathbb{R}^n)$. Then for any $\varphi \in C_0^{\infty}(B_r)$, $\Lambda \geq 5$, $r \in (0, R)$*

$$\|H(w, \varphi)\|_{\overline{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}} \leq C (2^k \Lambda)^{-\gamma} \|\Delta^{\frac{\alpha}{2}} \varphi\|_{\overline{p}}.$$

Proof. We have on $B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}$

$$H(w, \varphi) = \Delta^{\frac{\alpha}{2}}(w\varphi) - w\Delta^{\frac{\alpha}{2}}\varphi.$$

One checks that (exploiting the disjoint support via similar arguments as in Proposition 2.4)

$$\|\Delta^{\frac{\alpha}{2}}(w\varphi)\|_{\overline{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}} \prec \|w\|_{\infty} (2^k \Lambda)^{-n}.$$

and as well

$$\|w \Delta^{\frac{\alpha}{2}} \varphi\|_{\overline{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}} \prec \|w\|_{\infty} (2^k \Lambda)^{-n}.$$

□

Moreover,

Proposition 2.6. *There is a constant $C_{\overline{\alpha}}$ and an exponent $\gamma > 0$ such that the following holds. Let $\|w\|_{\infty} \leq 1$, and assume $\Delta^{\frac{\alpha}{2}} w \in L^{\overline{p}}(\mathbb{R}^n)$. Then for any $\varphi \in C_0^{\infty}(B_r)$, $\Lambda \geq 5$*

$$\|H(w, \varphi)\|_{\overline{p}, B_{\Lambda r}} \leq C \left(\Lambda^{-\gamma} \|\Delta^{\frac{\alpha}{2}} w\|_{\overline{p}} + \|\eta_{\Lambda^3 r} \Delta^{\frac{\alpha}{2}} w\|_{\overline{p}} \right) \|\Delta^{\frac{\alpha}{2}} \varphi\|_{\overline{p}}.$$

Proof. Set

$$w = \Delta^{-\frac{\alpha}{2}}(\eta_{\Lambda^3 r} \Delta^{\frac{\alpha}{2}} w) + \Delta^{-\frac{\alpha}{2}}((1 - \eta_{\Lambda^3 r}) \Delta^{\frac{\alpha}{2}} w) =: w_1 + w_2,$$

and

$$\varphi = \Delta^{-\frac{\alpha}{2}}(\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} \varphi) + \Delta^{-\frac{\alpha}{2}}((1 - \eta_{\Lambda r}) \Delta^{\frac{\alpha}{2}} \varphi) =: \varphi_1 + \varphi_2,$$

Then

$$H(w, \varphi) = H(w_1, \varphi) + H(w_2, \varphi_2) + H(w_2, \varphi_1).$$

We compute via Proposition A.7,

$$\begin{aligned} \|H(w_1, \varphi)\|_{\bar{p}} &\prec \|\eta_{\Lambda^{3r}} \Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}} \|\Delta^{\frac{\bar{\alpha}}{2}} \varphi\|_{\bar{p}}, \\ \|H(w_2, \varphi_2)\|_{\bar{p}} &\prec \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}} \left\| (1 - \eta_{\Lambda^r}) \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right\|_{\bar{p}} \prec \Lambda^{-n} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}} \left\| \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right\|_{\bar{p}}. \end{aligned}$$

In order to estimate $\|H(w_2, \varphi_1)\|_{\bar{p}}$ in the given set, according to Lemma A.6, similar to the arguments in [Sch11], it suffices to estimate terms of the following form, for some $\psi \in C_0^\infty(B_r)$, $\|\psi\|_{\bar{p}} \leq 1$, and for $s \in (0, \bar{\alpha})$, $t \in [0, s]$ (in fact, there might appear additional 0-multipliers, but as they do not interfere with the argument, we ignore this for the sake of readability)

$$\begin{aligned} &\int \Delta^{-\frac{s-t}{2}} \psi \Delta^{-\frac{t}{2}} \left| (1 - \eta_{\Lambda^{3r}}) \Delta^{\frac{\bar{\alpha}}{2}} w \right| \Delta^{-\frac{\bar{\alpha}-t}{2}} \left| \eta_{\Lambda^r} \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right| \\ = &\int \Delta^{-\frac{s-t}{2}} \psi \eta_{\Lambda^{2r}} \Delta^{-\frac{t}{2}} \left| (1 - \eta_{\Lambda^{3r}}) \Delta^{\frac{\bar{\alpha}}{2}} w \right| \Delta^{-\frac{\bar{\alpha}-s}{2}} \left| \eta_{\Lambda^r} \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right| \\ &+ \int (1 - \eta_{\Lambda^{2r}}) \Delta^{-\frac{s-t}{2}} \psi \Delta^{-\frac{t}{2}} \left| (1 - \eta_{\Lambda^{3r}}) \Delta^{\frac{\bar{\alpha}}{2}} w \right| \Delta^{-\frac{\bar{\alpha}-s}{2}} \left| \eta_{\Lambda^r} \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right| \\ \prec &\left\| \eta_{\Lambda^{2r}} \Delta^{-\frac{t}{2}} \left| (1 - \eta_{\Lambda^{3r}}) \Delta^{\frac{\bar{\alpha}}{2}} w \right| \right\|_{\frac{n}{\bar{\alpha}-t}} \left\| \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right\|_{\bar{p}} \\ &+ \left\| (1 - \eta_{\Lambda^{2r}}) \Delta^{-\frac{s-t}{2}} \psi \right\|_{\frac{n}{n-s+t-\bar{\alpha}}} \left\| \Delta^{\frac{\bar{\alpha}}{2}} w \right\|_{\bar{p}} \left\| \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right\|_{\bar{p}} \\ \prec &(\Lambda^{t-\bar{\alpha}} + \Lambda^{-\bar{\alpha}}) \left\| \Delta^{\frac{\bar{\alpha}}{2}} w \right\|_{\bar{p}} \left\| \Delta^{\frac{\bar{\alpha}}{2}} \varphi \right\|_{\bar{p}} \end{aligned}$$

Here, we have used several times the arguments for products of (non-local) fractional operators with disjoint support, for the details of which we refer to, e.g., the arguments of Proposition 2.4 or [Sch11, Proposition 4.4]. Setting $\gamma := \min\{\bar{\alpha} - t, \bar{\alpha}, n\}$, we conclude. \square

Conclusion for the tangential part

By Proposition 2.3, Proposition 2.5 and Proposition 2.6, we arrive at the following

Proposition 2.7 (Estimate of (2.8)). *There is $\gamma = \gamma_{\bar{\alpha}, \bar{p}} > 0$ and a constant C depending on the choice of $B, \tilde{B}, \bar{\eta}$, such that the following holds: For any $\varepsilon > 0$ there exists $\Lambda > 0, R > 0$, such that for any $B_r \subset\subset B, r \in (0, R)$, $\varphi \in C_0^\infty(B_r)$, $\|\Delta^{\frac{\bar{\alpha}}{2}} \varphi\|_{\bar{p}} \leq 1$, $\omega_{ij} = -\omega_{ji}$, $|\omega| \leq 1$,*

$$\omega_{ij} \int_{\mathbb{R}^n} |\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i \Delta^{\frac{\bar{\alpha}}{2}} \varphi \leq \varepsilon \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{\Lambda r}}^{\bar{p}-1} + C r^\gamma + \varepsilon \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}}^{\bar{p}-1}.$$

Now we can proceed by virtually the same arguments as in [DLR11b], [Sch12]: Firstly, Proposition 2.7 finally implies

Lemma 2.8. *There is $\gamma = \gamma_{\bar{\alpha}, \bar{p}} > 0$ and a constant C depending on the choice of $B, \tilde{B}, \bar{\eta}$, such that the following holds: For any $\varepsilon > 0$ there exists $\Lambda > 0, R > 0$, such that for any $B_r \subset\subset B, r \in (0, R)$, $\omega_{ij} = -\omega_{ji}$, $|\omega| \leq 1$,*

$$\left\| |\Delta^{\frac{\bar{\alpha}}{2}} w|^{\bar{p}-2} \omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i \right\|_{\bar{p}', B_{\Lambda^{-1} r}} \leq \varepsilon \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{\Lambda r}}^{\bar{p}-1} + C r^\gamma + \varepsilon \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}}^{\bar{p}-1}.$$

Putting tangential and normal part together

Together, Lemma 2.8 and Lemma 2.2 imply

Lemma 2.9. *There is $\gamma = \gamma_{\bar{\alpha}, \bar{p}} > 0$ and a constant C depending on the choice of $B, \tilde{B}, \bar{\eta}$, such that the following holds: For any $\varepsilon > 0$ there exists $\Lambda > 0, R > 0$, such that for any $B_r \subset\subset B, r \in (0, R)$,*

$$\|\Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1} r}}^{\bar{p}-1} \leq \varepsilon \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{\Lambda r}}^{\bar{p}-1} + C r^\gamma + \varepsilon \sum_{k=1}^{\infty} 2^{-\gamma k} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}}^{\bar{p}-1}.$$

Proof. Note that, if $\bar{p} \geq 2$,

$$\|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1} = \|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|^{\bar{p}-2} \omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}', B_{\Lambda^{-1}r}} \prec \|\Delta^{\frac{\bar{\alpha}}{2}} w\|^{\bar{p}-2} \omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}', B_{\Lambda^{-1}r}}$$

On the other hand, if $\bar{p} \in (1, 2)$, we set for $\theta > 0$ from Proposition B.1,

$$A_{\omega, \theta} := \left\{ |w^i \omega_{ij} \Delta^{\frac{\bar{\alpha}}{2}} w^j| \geq \theta |\Delta^{\frac{\bar{\alpha}}{2}} w| \right\}. \quad (2.11)$$

Said Proposition B.1 then implies that for any $\omega \in \Omega$,

$$\begin{aligned} & \|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1} \\ & \approx \|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r} \cap A_{\omega, \theta}}^{\bar{p}-1} + \|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r} \cap A_{\omega, \theta}^c}^{\bar{p}-1} \\ & \stackrel{P.B.1}{\prec} \|\omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r} \cap A_{\omega, \theta}}^{\bar{p}-1} + \sum_{\tilde{\omega} \in \Omega} \|\tilde{\omega}_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r} \cap A_{\tilde{\omega}, \theta}}^{\bar{p}-1} + \|w^i \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1} \\ & \stackrel{(2.11)}{\prec} C_{\theta} \sum_{\tilde{\omega} \in \Omega} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|^{\bar{p}-2} \tilde{\omega}_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}', B_{\Lambda^{-1}r}} + \|w^i \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1}. \end{aligned}$$

Applying again Proposition B.1, we arrive for any $\bar{p} > 1$ at

$$\|\Delta^{\frac{\bar{\alpha}}{2}} w\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1} \prec \sum_{\omega \in \Omega} \|\Delta^{\frac{\bar{\alpha}}{2}} w\|^{\bar{p}-2} \omega_{ij} w^j \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}', B_{\Lambda^{-1}r}} + \|w^i \Delta^{\frac{\bar{\alpha}}{2}} w^i\|_{\bar{p}, B_{\Lambda^{-1}r}}^{\bar{p}-1}.$$

We conclude by Lemma 2.8 and Lemma 2.2. \square

The proof of Theorem 2.1 follows by an iteration argument and an application of Dirichlet's growth theorem (cf. [DLR11b], [Sch11]).

A Lower Order Arguments: Proof of Theorem 1.2

Let $\alpha \in (0, n)$. In this section, we treat estimates on the bilinear operator

$$H_\alpha(u, v) := \Delta^{\frac{\alpha}{2}}(uv) - u\Delta^{\frac{\alpha}{2}}v - v\Delta^{\frac{\alpha}{2}}u,$$

which behaves, in some sense, like a product of lower operators (take, for example, the classic case $\alpha = 2$). The estimates are similar to the ones obtained and used in [DLR11b], [Sch12], [DLR11a], [DL10], [Sch11], and here, we will adopt the general strategy of the latter article [Sch11]. The argument relies on the estimates on singular kernels of Proposition A.1. These kernels (the geometric-space analogue to the Fourier-multiplier estimates in [Sch12]) appear because of the following representation of $\Delta^{\frac{\alpha}{2}}$, $\Delta^{-\frac{\alpha}{2}}$: For $f \in C_0^\infty(\mathbb{R}^n)$,

$$\Delta^{\frac{\alpha}{2}}f(x) = c_{\alpha,n} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dx, \quad \text{if } \alpha \in (0, 1).$$

The inverse operator of $\Delta^{\frac{\alpha}{2}}$ is the so-called Riesz potential,

$$\Delta^{-\frac{\alpha}{2}}f(x) = c_{\alpha,n} \int_{\mathbb{R}^n} f(\eta) |x - \eta|^{-n+\alpha} d\eta, \quad \text{if } \alpha \in (0, n). \quad (\text{A.1})$$

The essential argument appears in the case $\alpha \in (0, 1)$. As mentioned above, they are similar to the ones in [Sch11], though there, for convenience, only special cases for α were considered.

A.1 The case $\alpha \in (0, 1)$

With the representation for $\Delta^{\frac{\alpha}{2}}$, one has for $\alpha \in (0, 1)$,

$$\begin{aligned} \Delta^{\frac{\alpha}{2}}(u v)(x) &= c_{\alpha,n} \int \frac{u(x) v(x) - u(y) v(y)}{|x - y|^{n+\alpha}} dy \\ &= c_{\alpha,n} \int \frac{(u(x) - u(y)) v(x) + u(y) (v(x) - v(y))}{|x - y|^{n+\alpha}} dy \\ &= \Delta^{\frac{\alpha}{2}}u(x) v(x) + c_{\alpha,n} \int \frac{(u(y) - u(x)) (v(x) - v(y))}{|x - y|^{n+\alpha}} dy + \Delta^{\frac{\alpha}{2}}b(x) u(x), \end{aligned}$$

that is

$$H_\alpha(u, v) = c_{\alpha,n} \int \frac{(u(y) - u(x)) (v(x) - v(y))}{|x - y|^{n+\alpha}} dy.$$

Replacing now $u := \Delta^{-\frac{\alpha}{2}}a$, $v := \Delta^{-\frac{\alpha}{2}}b$, this is equivalent to

$$H_\alpha(u, v) = \tilde{c}_{\alpha,n} \int \int \int \frac{(|y - \eta|^{-n+\alpha} - |x - \eta|^{-n+\alpha}) (|y - \xi|^{-n+\alpha} - |x - \xi|^{-n+\alpha})}{|x - y|^{n+\alpha}} a(\eta) b(\xi) d\xi d\eta dy.$$

Thus, the main point of our argument is to replace the differences of functions in the definition of H by differences on the kernels of the respective operators. In the above representation of $H_\alpha(u, v)$ it is useful to observe: whenever $|x - y|^{-n+\alpha}$ becomes singular, both, the difference of terms with η and the terms with ξ tend to zero as well, thus “absorbing” the singularity up to a certain point. More precisely, these kernels can be estimated as in the following proposition.

Proposition A.1 (Multiplier estimate). *Let $\alpha \in [0, 1]$ and $\varepsilon \in (0, 1)$. Then for almost every $x, y, \eta, \xi \in \mathbb{R}^n$, and a uniform constant C*

$$\begin{aligned} & \left| |y - \eta|^{-n+\alpha} - |\eta - x|^{-n+\alpha} \right| \left| |\xi - y|^{-n+\alpha} - |\xi - x|^{-n+\alpha} \right| \\ & \leq C |y - \eta|^{-n+\alpha-\varepsilon} \left(|x - \xi|^{-n+\alpha-\varepsilon} + |y - \xi|^{-n+\alpha-\varepsilon} \right) |x - y|^{2\varepsilon} \\ & \quad + C \left(|x - \eta|^{-n+\alpha-\varepsilon} + |y - \eta|^{-n+\alpha-\varepsilon} \right) |y - \xi|^{-n+\alpha-\varepsilon} |x - y|^{2\varepsilon} \\ & \quad + C |x - \eta|^{-n+\alpha} |x - \xi|^{-n+\alpha} \chi_{|x-y|>2|x-\xi|} \chi_{|x-y|>2|x-\eta|}. \end{aligned}$$

In particular, multiplying this estimate with the hypersingular (i.e. not locally integrable) kernel $|x - y|^{-n-\alpha}$ (the kernel of the differentiation $\Delta^{\frac{\alpha}{2}}$), choosing ε such that $1 > 2\varepsilon > \alpha > \varepsilon > 0$, the first two terms on the right-hand side consists of products of locally integrable kernels, or, more precisely, kernels of the Riesz potentials $\Delta^{-\frac{\alpha-\varepsilon}{2}}$ and $\Delta^{-\frac{2\varepsilon-\alpha}{2}}$. In the last term, we then have twice kernels of $\Delta^{-\frac{\alpha}{2}}$, and the kernel $|x - y|^{-n-\alpha} \chi_{|x-y|>2|x-\xi|} \chi_{|x-y|>2|x-\eta|}$ where the singularity $x = y$ is somewhat cut away.

As we will see in Proposition A.2, this enables us, to show that the operators $\Delta^{\frac{\alpha}{2}}$ in the definition of $H_\alpha(\cdot, \cdot)$ “distribute their differentiation” on both entry-functions.

Proof of Proposition A.1. Observe that in the following argument, since $\alpha \in [0, 1]$, all the constants can be taken independently from α . We set

$$k(x, y, \eta) := \left| |\eta - y|^{-n+\alpha} - |\eta - x|^{-n+\alpha} \right|.$$

Decompose the space $(x, y, \eta) \in \mathbb{R}^{3n}$ into several subspaces depending on the relations of $|y - \eta|$, $|x - y|$, $|x - \eta|$:

$$1 \leq \chi_1(x, y, \eta) + \chi_2(x, y, \eta) + \chi_3(x, y, \eta) \quad \text{for } x, y, \eta \in \mathbb{R}^n,$$

where

$$\chi_1 := \chi_{|x-y| \leq 2|y-\eta|} \chi_{|x-y| \leq 2|x-\eta|},$$

$$\chi_2 := \chi_{|x-y| \leq 2|y-\eta|} \chi_{|x-y| > 2|x-\eta|},$$

$$\chi_3 := \chi_{|x-y| > 2|y-\eta|} \chi_{|x-y| \leq 2|x-\eta|},$$

and functions of the form $\chi_{f(x,y,\eta) < 0}$ denote the usual characteristic functions of the set $\{(x, y, \eta) \in \mathbb{R}^{3N} : f(x, y, \eta) < 0\}$. Note that with a uniform constant

$$|y - \eta| \chi_1 \approx |x - \eta| \chi_1. \quad (\text{A.2})$$

Then, by the mean value theorem, for the details in this context cf. [Sch11, Proposition 3.3], for any $\varepsilon \in (0, 1)$

$$k(x, y, \eta) \chi_1 \prec |x - \eta|^{-n+\alpha-1} |x - y| \chi_1 \prec |x - \eta|^{-n+\alpha-\varepsilon} |x - y|^\varepsilon \chi_1 \stackrel{(\text{A.2})}{\approx} |y - \eta|^{-n+\alpha-\varepsilon} |x - y|^\varepsilon \chi_1.$$

Moreover, for any $\varepsilon > 0$,

$$k(x, y, \eta) \chi_2 \prec |x - \eta|^{-n+\alpha} \chi_2 \prec |x - \eta|^{-n+\alpha-\varepsilon} |x - y|^\varepsilon,$$

and also for any $\varepsilon > 0$,

$$k(x, y, \eta) \chi_3 \prec |y - \eta|^{-n+\alpha} \chi_3 \prec |y - \eta|^{-n+\alpha-\varepsilon} |x - y|^\varepsilon.$$

In order to estimate the product $k(x, y, \xi) k(x, y, \eta)$ we have to check the claim for all cases (i, j) , $i, j \in \{1, 2, 3\}$, where we say

$$\text{case } (i, j) \Leftrightarrow (x, y, \eta, \xi) \in \mathbb{R}^{4n} \text{ such that } \chi_i(x, y, \eta) \chi_j(x, y, \xi) = 1.$$

Lets denote

$$\text{Type I-estimate} := |y - \eta|^{-n+\alpha-\varepsilon} |x - \xi|^{-n+\alpha-\varepsilon} |x - y|^{2\varepsilon}$$

$$\text{Type II-estimate} := |y - \eta|^{-n+\alpha-\varepsilon} |y - \xi|^{-n+\alpha-\varepsilon} |x - y|^{2\varepsilon}$$

$$\text{Type III-estimate} := |x - \eta|^{-n+\alpha-\varepsilon} |y - \xi|^{-n+\alpha-\varepsilon} |x - y|^{2\varepsilon}$$

$$\text{Type IV-estimate} := |x - \eta|^{-n+\alpha} |x - \xi|^{-n+\alpha} \chi_{|x-y|>2|x-\xi|} \chi_{|x-y|>2|x-\eta|}.$$

One checks, that each of these types have to appear. Note, the only case where we choose the Type IV-estimate is $(2, 2)$. \square

As a consequence of Proposition A.1, we obtain the following estimate for $\alpha \in (0, 1)$:

Proposition A.2. *Let $u = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} u$, $v = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} v$. Then for $\alpha \in (0, 1)$ and for $\varepsilon \in (0, 1)$ satisfying*

$$\alpha < 2\varepsilon < \min\{2\alpha, n + \alpha\},$$

there exists $C \equiv C_\alpha > 0$ such that

$$\begin{aligned}
& |H(\Delta^{-\frac{\alpha}{2}}a, \Delta^{-\frac{\alpha}{2}}b)(x)| \\
& \leq C[\Delta^{-\frac{\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}u|(x) \Delta^{-\frac{\alpha-\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}v|(x) + \Delta^{-\frac{\alpha-\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}u|(x) \Delta^{-\frac{\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}v|(x) \\
& \quad \Delta^{-\frac{2\varepsilon-\alpha}{2}}\left(\Delta^{-\frac{\alpha-\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}u| \Delta^{-\frac{\alpha-\varepsilon}{2}}|\Delta^{\frac{\alpha}{2}}v|\right)(x) \\
& \quad + \Delta^{-\frac{\alpha}{4}}|\Delta^{\frac{\alpha}{2}}u|(x) \Delta^{-\frac{\alpha}{4}}|\Delta^{\frac{\alpha}{2}}v|(x)].
\end{aligned}$$

Proof. Set $a := \Delta^{\frac{\alpha}{2}}u$, $b := \Delta^{\frac{\alpha}{2}}v$. By the definition of $\Delta^{\frac{\alpha}{2}}$ for $\alpha \in (0, 1)$,

$$\begin{aligned}
& \Delta^{\frac{\alpha}{2}}(\Delta^{-\frac{\alpha}{2}}a \Delta^{-\frac{\alpha}{2}}b)(x) \\
& = c_{n,\alpha} \int \frac{\Delta^{-\frac{\alpha}{2}}a(x) \Delta^{-\frac{\alpha}{2}}b(x) - \Delta^{-\frac{\alpha}{2}}a(y) \Delta^{-\frac{\alpha}{2}}b(y)}{|x-y|^{n+\alpha}} dy \\
& = c_{n,\alpha} \int \frac{(\Delta^{-\frac{\alpha}{2}}a(x) - \Delta^{-\frac{\alpha}{2}}a(y)) \Delta^{-\frac{\alpha}{2}}b(x) + \Delta^{-\frac{\alpha}{2}}a(y) (\Delta^{-\frac{\alpha}{2}}b(x) - \Delta^{-\frac{\alpha}{2}}b(y))}{|x-y|^{n+\alpha}} dy \\
& = a(x) \Delta^{-\frac{\alpha}{2}}b(x) + c_{n,\alpha} \int \frac{(\Delta^{-\frac{\alpha}{2}}a(y) - \Delta^{-\frac{\alpha}{2}}a(x)) (\Delta^{-\frac{\alpha}{2}}b(x) - \Delta^{-\frac{\alpha}{2}}b(y))}{|x-y|^{n+\alpha}} dy + b(x) \Delta^{-\frac{\alpha}{2}}a(x).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& |H(\Delta^{-\frac{\alpha}{2}}a, \Delta^{-\frac{\alpha}{2}}b)(x)| \\
& \prec \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{||y-\eta|^{-n+\alpha} - |x-\eta|^{-n+\alpha}| \, ||y-\xi|^{-n+\alpha} - |x-\xi|^{-n+\alpha}|}{|x-y|^{n+\alpha}} |a|(\eta) |b|(\xi) dy d\xi d\eta.
\end{aligned}$$

Pick $\varepsilon \in (0, 1)$ in Proposition A.1 such that

$$\alpha < 2\varepsilon < \min\{2\alpha, n + \alpha\}.$$

Let us see, for instance, how our expression behaves if the “Type I”-estimate from the proof of Proposition A.1 is applicable, that is, if

$$\frac{||y-\eta|^{-n+\alpha} - |x-\eta|^{-n+\alpha}| \, ||y-\xi|^{-n+\alpha} - |x-\xi|^{-n+\alpha}|}{|x-y|^{n+\alpha}} \prec |y-\eta|^{-n+\alpha-\varepsilon} |x-\xi|^{-n+\alpha-\varepsilon} |x-y|^{-n+2\varepsilon-\alpha}$$

Observe, by the choice of ε , all the appearing kernels on the right-hand side of this estimate have the exponent $n - \sigma$ for some $\sigma > 0$. That is, all the appearing kernels correspond to the kernel of a Riesz potential $\Delta^{-\frac{\sigma}{2}}$, see (A.1). Namely,

$$\begin{aligned}
& \int |y-\eta|^{-n+\alpha-\varepsilon} |a|(\eta) d\eta \approx \Delta^{-\frac{\alpha-\varepsilon}{2}}|a|(y), \\
& \int |x-\xi|^{-n+\alpha-\varepsilon} |b|(\xi) d\xi \approx \Delta^{-\frac{\alpha-\varepsilon}{2}}|b|(x),
\end{aligned}$$

and finally

$$\int |x-y|^{-n+2\varepsilon-\alpha} \Delta^{-\frac{\alpha-\varepsilon}{2}}|a|(y) dy \approx \Delta^{-\frac{2\varepsilon-\alpha}{2}} \Delta^{-\frac{\alpha-\varepsilon}{2}}|a|(x) \approx \Delta^{-\frac{\varepsilon}{2}}|a|(x).$$

By these kind of arguments, one obtains

$$\begin{aligned}
& |H(\Delta^{-\frac{\alpha}{2}}a, \Delta^{-\frac{\alpha}{2}}b)(x)| \\
& \prec \Delta^{-\frac{\varepsilon}{2}}|a|(x) \Delta^{-\frac{\alpha-\varepsilon}{2}}|b|(x) + \Delta^{-\frac{\alpha-\varepsilon}{2}}|a|(x) \Delta^{-\frac{\varepsilon}{2}}|b|(x) \\
& \quad \Delta^{-\frac{2\varepsilon-\alpha}{2}}\left(\Delta^{-\frac{\alpha-\varepsilon}{2}}|a| \Delta^{-\frac{\alpha-\varepsilon}{2}}|b|\right)(x) \\
& \quad + A,
\end{aligned}$$

where

$$\begin{aligned}
A &:= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|x-\eta|^{-n+\alpha} |x-\xi|^{-n+\alpha}}{|x-y|^{n+\alpha}} \chi_{|x-y|>2|x-\xi|} \chi_{|x-y|>2|x-\eta|} |a|(\eta) |b|(\xi) dy d\xi d\eta \\
&\prec \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-\eta|^{-n+\alpha} |x-\xi|^{-n+\alpha} |x-\xi|^{-\frac{\alpha}{2}} |x-\eta|^{-\frac{\alpha}{2}} |a|(\eta) |b|(\xi) d\xi d\eta \\
&= \Delta^{-\frac{\alpha}{4}} |a|(x) \Delta^{-\frac{\alpha}{4}} |b|(x).
\end{aligned}$$

Thus we can conclude the proof: we choose $L_\alpha = 3$, with $s_1 = \varepsilon$, $t_1 = \alpha - \varepsilon$, $s_1 - t_1 = 2\varepsilon - \alpha$ as first term, then with interchanged roles $s_2 = t_1$ and $t_2 = s_2$, $s_2 - t_2 = 2\varepsilon - \alpha$ and finally $s_3 = \alpha/2$, $t_3 = \alpha/2$.

□

Remark A.3. About the strategy of the above proof let us remark, how the decomposition argument we apply in [Sch11] and here, is related to the arguments in [DLR11b] and [Sch12]: For the para-product estimates of the 3-term commutators in [DLR11b], the authors used an infinite Taylor expansion after Fourier transform in the phase space whenever the symbols were rather close to each other.

Instead, in [Sch12] the mean value formula (that is: a one-step Taylor expansion) was employed, also after Fourier transform, which lead to a simple pointwise estimate in the phase space which sufficed for the purposes there, but did not have the full power of the more complicated, yet more generalizable argument in [DLR11b]. Both of these arguments, essentially obtained *pointwise* bounds in the *phase space*, which – transformed in the *geometric space* – gives not pointwise, but $L^p(\mathbb{R}^n)$ -estimates.

With the method introduced in [Sch11] and employed here, we obtain pointwise results in the *geometric space*.

But let us stress, that both, the arguments in [Sch12] in the phase space and in [Sch11] in the geometric space, which apply both a one-step Taylor expansion, are rough in the following sense: When considering Hardy-space or BMO-space estimates, they do not seem to give the optimal result (which is, anyways, not pointwise anymore), and in this case, the strategy of using para-products as developed in [DLR11b] seems more viable, see, e.g., the Hardy-space estimates in [DLR11b].

A.2 The case $\alpha \geq 1$

We will reduce the case $\alpha \geq 1$ to the case $\alpha \in (0, 1)$ already discussed. For the case $\alpha = 1 + \tilde{\alpha} \in [1, 2)$, we use the following argument: Let \mathcal{R}_i be the i -th Riesz-transform,

$$\mathcal{R}_i f(x) = c \int \frac{(x-y)^i}{|x-y|^{n+1}} f(y) dy.$$

Then for $1 + \tilde{\alpha} \in [1, 2)$,

$$\begin{aligned}
\mathcal{R}_i H_{1+\tilde{\alpha}}(u, v) &= \Delta^{\frac{\tilde{\alpha}}{2}} \partial_i(uv) - \mathcal{R}_i(u \Delta^{\frac{1+\tilde{\alpha}}{2}} v) - \mathcal{R}_i(v \Delta^{\frac{1+\tilde{\alpha}}{2}} u) \\
&= \Delta^{\frac{\tilde{\alpha}}{2}} (u \partial_i v) - u \Delta^{\frac{\tilde{\alpha}}{2}} \partial_i v - \Delta^{\frac{\tilde{\alpha}}{2}} u \partial_i v \\
&\quad + \Delta^{\frac{\tilde{\alpha}}{2}} (v \partial_i u) - v \Delta^{\frac{\tilde{\alpha}}{2}} \partial_i u - \Delta^{\frac{\tilde{\alpha}}{2}} v \partial_i u \\
&\quad - \mathcal{R}_i(u \Delta^{\frac{1+\tilde{\alpha}}{2}} v) + u \Delta^{\frac{\tilde{\alpha}}{2}} \partial_i v \\
&\quad - \mathcal{R}_i(v \Delta^{\frac{1+\tilde{\alpha}}{2}} u) + v \Delta^{\frac{\tilde{\alpha}}{2}} \partial_i u \\
&\quad + \Delta^{\frac{\tilde{\alpha}}{2}} v \partial_i u + \Delta^{\frac{\tilde{\alpha}}{2}} u \partial_i v \\
&= H_{\tilde{\alpha}}(u, \partial_i v) + H_{\tilde{\alpha}}(v, \partial_i u) \\
&\quad - \mathcal{R}_i(u \Delta^{\frac{1+\tilde{\alpha}}{2}} v) + u \mathcal{R}_i \Delta^{\frac{1+\tilde{\alpha}}{2}} v \\
&\quad - \mathcal{R}_i(v \Delta^{\frac{1+\tilde{\alpha}}{2}} u) + v \mathcal{R}_i \Delta^{\frac{1+\tilde{\alpha}}{2}} u \\
&\quad + \Delta^{\frac{\tilde{\alpha}}{2}} v \partial_i u + \Delta^{\frac{\tilde{\alpha}}{2}} u \partial_i v.
\end{aligned}$$

In the case $\tilde{\alpha} = 0$, we write this as

$$\begin{aligned}
\mathcal{R}_i H_1(u, v) &= -\mathcal{R}_i(u \Delta^{\frac{1+\tilde{\alpha}}{2}} v) + u \mathcal{R}_i \Delta^{\frac{1+\tilde{\alpha}}{2}} \partial_i v \\
&\quad - \mathcal{R}_i(v \Delta^{\frac{1+\tilde{\alpha}}{2}} u) + v \mathcal{R}_i \Delta^{\frac{1+\tilde{\alpha}}{2}} u
\end{aligned}$$

The terms $H_{\tilde{\alpha}}(\cdot, \cdot)$, as we've seen above, and the terms $\Delta^{\frac{\tilde{\alpha}}{2}} u \partial_i v$ and $\Delta^{\frac{\tilde{\alpha}}{2}} v \partial_i u$ are actually products of lower order operators applied to u and v , respectively. Moreover, we have the following estimate, which should be compared to the famous commutator estimates and their relation to Hardy spaces and BMO, by Coifman, Rochberg, Weiss [CRW76] and the related work by [Cha82].

Proposition A.4. *For $\tilde{\alpha} \in [0, 1)$ there exist a constant $C_{\tilde{\alpha}} > 0$ and a number $L \equiv L_{\tilde{\alpha}} \in \mathbb{N}$, and for $k \in \{1, \dots, L\}$ constants $s_k \in (0, 1)$, $t_k \in [0, s_k]$ such that*

$$\mathcal{R}_i(G \Delta^{-\frac{1+\tilde{\alpha}}{2}} F) - (\mathcal{R}_i G \Delta^{-\frac{1+\tilde{\alpha}}{2}} F) \prec C \sum_{k=1}^L \Delta^{-\frac{s_k - t_k}{2}} \left(\Delta^{-\frac{t_k}{2}} |G| \Delta^{-\frac{1+\tilde{\alpha}}{2} + \frac{s_k}{2}} |F| \right).$$

Moreover, $|s_k - t_k|$ can be supposed to be arbitrarily small.

Proof. As in the case of H_{α} for $\alpha \in (0, 1)$ above, we exploit that the difference of the involved operators can be expressed by the differences of their kernels, more precisely

$$\begin{aligned}
&\mathcal{R}_i(G \Delta^{-\frac{1+\tilde{\alpha}}{2}} F)(x) - (\mathcal{R}_i G \Delta^{-\frac{1+\tilde{\alpha}}{2}} F)(x) \\
&= c \int \int \frac{(x-y)^i}{|x-y|^{n+1}} (|y-z|^{-n+1+\tilde{\alpha}} - |x-z|^{-n+1+\tilde{\alpha}}) G(y) F(z) dy dz \\
&\prec c \int \int \frac{|y-z|^{-n+1+\tilde{\alpha}} - |x-z|^{-n+1+\tilde{\alpha}}}{|x-y|^n} |G|(y) |F|(z) dy dz
\end{aligned}$$

Using the mean value theorem, similar to the proof of Proposition A.1, precisely as in [Sch11], we conclude. \square

In particular, we have

Proposition A.5. *Let $u = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} u$, $v = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} v$. Then for $\alpha \in (0, 2)$ there exists some constant $C_\alpha > 0$ and a number $L \equiv L_\alpha \in \mathbb{N}$, and for $k \in \{1, \dots, L\}$ constants $s_k \in (0, \alpha)$, $t_k \in [0, s_k]$ such that for any $i = 1, \dots, n$, where \mathcal{R}_i denotes the Riesz transform,*

$$|\mathcal{R}_i H_\alpha(u, v)(x)| \leq C \sum_{k=1}^L \Delta^{-\frac{s_k - t_k}{2}} \left(M_k \Delta^{-\frac{t_k}{2}} |\Delta^{\frac{\alpha}{2}} u| \ N_k \Delta^{-\frac{\alpha}{2} + \frac{s_k}{2}} |\Delta^{\frac{\alpha}{2}} v| \right).$$

Here, M_k, N_k are possibly Riesz transforms, or the identity. Moreover, $|s_k - t_k|$ can be supposed to be arbitrarily small.

For $\alpha = K + \tilde{\alpha}$, $\tilde{\alpha} \in (0, 2)$, $K \in \mathbb{N}$, observe that

$$\begin{aligned} H_\alpha(u, v) &= \Delta^{\frac{\tilde{\alpha}}{2}} (\Delta^K u \ v) + \Delta^{\frac{\tilde{\alpha}}{2}} (u \ \Delta^K v) + \sum_{\substack{|\gamma| + |\tilde{\gamma}| = 2K \\ |\gamma|, |\tilde{\gamma}| \geq 1}} c_{\gamma, \tilde{\gamma}} \Delta^{\frac{\tilde{\alpha}}{2}} (\partial^\gamma u \partial^{\tilde{\gamma}} v) \\ &\quad - u \ \Delta^K \Delta^{\frac{\tilde{\alpha}}{2}} v - u \Delta^K \Delta^{\frac{\tilde{\alpha}}{2}} v \\ &= H_{\tilde{\alpha}}(\Delta^K u, v) + \Delta^K u \ \Delta^{\frac{\tilde{\alpha}}{2}} v + H_{\tilde{\alpha}}(u, \Delta^K v) + \Delta^K v \ \Delta^{\frac{\tilde{\alpha}}{2}} u \\ &\quad + \sum_{\substack{|\gamma| + |\tilde{\gamma}| = 2K \\ |\gamma|, |\tilde{\gamma}| \geq 1}} c_{\gamma, \tilde{\gamma}} H_{\tilde{\alpha}}(\partial^\gamma u, \partial^{\tilde{\gamma}} v) \\ &\quad + \sum_{\substack{|\gamma| + |\tilde{\gamma}| = 2K \\ |\gamma|, |\tilde{\gamma}| \geq 1}} c_{\gamma, \tilde{\gamma}} \Delta^{\frac{\tilde{\alpha}}{2}} \partial^\gamma u \ \partial^{\tilde{\gamma}} v + \partial^\gamma u \ \Delta^{\frac{\tilde{\alpha}}{2}} \partial^{\tilde{\gamma}} v \end{aligned}$$

Using that all terms which are not of the form H_α , are actually products of lower order operators, one concludes

Lemma A.6. *Let $u = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} u$, $v = \Delta^{-\frac{\alpha}{2}} \Delta^{\frac{\alpha}{2}} v$. Then for $\alpha \in (0, n)$ there exists some constant $C_\alpha > 0$ and a number $L \equiv L_\alpha \in \mathbb{N}$, and for $k \in \{1, \dots, L\}$ constants $s_k \in (0, \alpha)$, $t_k \in [0, s_k]$ such that for any $i = 1, \dots, n$,*

$$|\mathcal{R}_i H_\alpha(u, v)(x)| \leq C \sum_{k=1}^L M_k \Delta^{-\frac{s_k - t_k}{2}} \left(\Delta^{-\frac{t_k}{2}} |\Delta^{\frac{\alpha}{2}} u| \ N_k \Delta^{-\frac{\alpha}{2} + \frac{s_k}{2}} |\Delta^{\frac{\alpha}{2}} v| \right).$$

Here, M_k, N_k are possibly Riesz transforms, or the identity. Moreover, $|s_k - t_k|$ can be supposed to be arbitrarily small.

In particular,

Proposition A.7. *Let $\alpha \in (0, n)$, $q, q_1, q_2 \in [1, \infty]$ such that*

$$\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|H_\alpha(u, v)\|_{(\frac{n}{\alpha}, q), \mathbb{R}^n} \prec \|\Delta^{\frac{\alpha}{2}} u\|_{(\frac{n}{\alpha}, q_2), \mathbb{R}^n} \|\Delta^{\frac{\alpha}{2}} v\|_{(\frac{n}{\alpha}, q_2), \mathbb{R}^n}.$$

A.3 Local Estimates

In this section, the goal is give in Lemma A.9 a localized version of Proposition A.7.

Similar to the arguments in Proposition 2.6, one can show

Proposition A.8. *There is $\gamma > 0$ such that for any $\Lambda > 4$, $s \in (0, \alpha)$, $t \in [0, s]$,*

$$\|\Delta^{-\frac{s-t}{2}} \left(\Delta^{-\frac{t}{2}} |\eta_{-\Lambda r} a| \ \Delta^{-\frac{\alpha}{2} + \frac{s}{2}} |b| \right)\|_{\frac{n}{\alpha}, B_r} \prec \Lambda^{-\gamma} \|a\|_{\frac{n}{\alpha}} \|b\|_{\frac{n}{\alpha}}$$

and for $\Lambda_1, \Lambda_2 > 4$

$$\|\Delta^{-\frac{s-t}{2}} \left(\Delta^{-\frac{t}{2}} |\eta_{-\Lambda_1 r} a| \ \Delta^{-\frac{\alpha}{2} + \frac{s}{2}} (\eta_{-\Lambda_2 r} |b|) \right)\|_{\frac{n}{\alpha}, B_r} \prec \Lambda_1^{-\gamma} \ \Lambda_2^{-\gamma} \|a\|_{\frac{n}{\alpha}} \|b\|_{\frac{n}{\alpha}}$$

Proof. For some $\psi \in C_0^\infty(B_r)$, $\|\psi\|_{\frac{n}{n-\alpha}} \leq 1$ we have to estimate (up to possibly Riesz transforms, which we will ignore again, as they do not change the argument)

$$\begin{aligned}
& \int \Delta^{-\frac{s-t}{2}} \left(\Delta^{-\frac{t}{2}} |\eta_{-\Lambda} a| \Delta^{-\frac{\alpha}{2} + \frac{s}{2}} |b| \right) \psi \\
&= \int \Delta^{-\frac{t}{2}} |\eta_{-\Lambda} a| \Delta^{-\frac{\alpha}{2} + \frac{s}{2}} |b| \Delta^{-\frac{s-t}{2}} \psi \\
&\prec \|\eta_{\sqrt{\Lambda}} \Delta^{-\frac{t}{2}} |\eta_{-\Lambda} a|\|_{\frac{n}{\alpha-t}} \|\Delta^{-\frac{\alpha}{2} + \frac{s}{2}} |b|\|_{\frac{n}{s}} \|\Delta^{-\frac{s-t}{2}} \psi\|_{\frac{n}{n-\alpha-s+t}} \\
&\quad + \|\Delta^{-\frac{t}{2}} |\eta_{-\Lambda} a|\|_{\frac{n}{\alpha-t}} \|\Delta^{-\frac{\alpha}{2} + \frac{s}{2}} |b|\|_{\frac{n}{s}} \|(1 - \eta_{\sqrt{\Lambda}}) \Delta^{-\frac{s-t}{2}} \psi\|_{\frac{n}{n-\alpha-s+t}} \\
&\prec \|\eta_{\sqrt{\Lambda}} \Delta^{-\frac{t}{2}} |\eta_{-\Lambda} a|\|_{\frac{n}{\alpha-t}} \|b\|_{\frac{n}{\alpha}} + \|a\|_{\frac{n}{\alpha}} \|b\|_{\frac{n}{\alpha}} \|(1 - \eta_{\sqrt{\Lambda}}) \Delta^{-\frac{s-t}{2}} \psi\|_{\frac{n}{n-\alpha-s+t}}
\end{aligned}$$

Then the first claim follows the arguments for products of (non-local) fractional operators with disjoint support, cf., e.g., the arguments of Proposition 2.4 or [Sch11, Proposition 4.4].

The second claim follows by the same method, only taking more care in the cutoff for b . \square

Lemma A.9. *Let $v, w \in W^{\alpha,p}(\mathbb{R}^n)$. Then for any $\Lambda > 2$*

$$\begin{aligned}
\|H(v, w)\|_{\frac{n}{\alpha}, B_r} &\prec \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} v\|_{\frac{n}{\alpha}} \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}} \\
&\quad + \Lambda^{-\gamma} \|\Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}} \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} v\|_{\frac{n}{\alpha}} \\
&\quad + \Lambda^{-\gamma} \|\Delta^{\frac{\alpha}{2}} v\|_{\frac{n}{\alpha}} \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}} \\
&\quad + \Lambda^{-\gamma} \|\Delta^{\frac{\alpha}{2}} v\|_{\frac{n}{\alpha}} \sum_{k=1}^{\infty} 2^{-k\gamma} \|\eta_{\Lambda r}^k \Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}}.
\end{aligned}$$

Proof. Set

$$\begin{aligned}
v_{\Lambda} &:= \Delta^{-\frac{\alpha}{2}} (\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} v), \\
v_{-\Lambda} &:= \Delta^{-\frac{\alpha}{2}} ((1 - \eta_{\Lambda r}) \Delta^{\frac{\alpha}{2}} v),
\end{aligned}$$

and

$$v_{-\Lambda, k} := \Delta^{-\frac{\alpha}{2}} (\eta_{\Lambda r}^k \Delta^{\frac{\alpha}{2}} v).$$

Then

$$H(v, w) = H(v_{\Lambda}, w_{\Lambda}) + H(v_{-\Lambda}, w_{\Lambda}) + H(v_{\Lambda}, w_{-\Lambda}) + H(v_{-\Lambda}, w_{-\Lambda}).$$

We have,

$$\|H(v_{\Lambda}, w_{\Lambda})\|_{\frac{n}{\alpha}} \prec \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} v\|_{\frac{n}{\alpha}} \|\eta_{\Lambda r} \Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}}.$$

Moreover, the second and the third term are controlled by means of Proposition A.8. It remains to estimate $H(v_{-\Lambda}, w_{-\Lambda})$,

$$H(v_{-\Lambda}, w_{-\Lambda}) = \sum_{k=1}^{\infty} H(v_{-\Lambda}, w_{-\Lambda, k})$$

and again by Proposition A.8,

$$\|H(v_{-\Lambda}, w_{-\Lambda, k})\|_{\frac{n}{\alpha}, B_r} \prec 2^{-\gamma k} \Lambda^{-2\gamma} \|v\|_{\frac{n}{\alpha}} \|\eta_{\Lambda r}^k \Delta^{\frac{\alpha}{2}} w\|_{\frac{n}{\alpha}}$$

\square

B Decomposition in Euclidean Spaces

In the proof of Lemma 2.9 we used the following fact, which permitted us to get a full information of $\|\Delta^{\frac{\alpha}{2}}w\|_{\overline{p}}$ from the information of $\|\omega_{ij}w^j\Delta^{\frac{\alpha}{2}}w^i\|_{\overline{p}}$, and the normal projection $\|w^i\Delta^{\frac{\alpha}{2}}w^i\|_{\overline{p}}$.

Proposition B.1. *Let $p \in \mathbb{R}^N$, $|p| = 1$. Set*

$$\Omega := \{\omega \equiv (\omega_{ij})_{i,j=1}^N : \omega_{ij} = -\omega_{ji} \in \{-1, 0, 1\}\},$$

and let for $\omega \in \Omega$

$$p_\omega \equiv (p_{\omega i})_{i=1}^N := (\omega_{ij}p^j)_{i=1}^N.$$

Then, for uniform constants c, C , depending only on the dimension N ,

$$c|q| \leq \sum_{\omega \in \Omega} |\langle p_\omega, q \rangle| + |\langle p, q \rangle| \leq C|q| \quad \text{for all } q \in \mathbb{R}^N. \quad (\text{B.1})$$

In particular, there is a uniform $\theta \in (0, 1)$ such that for any $q \in \mathbb{R}^N$, if for some $\omega \in \Omega$,

$$|\langle p_\omega, q \rangle| \leq \theta|q|,$$

then there exists $\tilde{\omega} \in \Omega$, such that

$$|\langle p_\omega, q \rangle| \leq \theta|q| \leq \max\{|\langle p_{\tilde{\omega}}, q \rangle|, |\langle p, q \rangle|\}.$$

For the convenience of the reader, we will give a proof:

Proof. We prove the first inequality of the claim (B.1) for $|q| = 1$ and $q \perp p$. The case $N = 1$ is trivial, of course, so assume $N \geq 2$. Let for $\alpha \neq \beta$,

$$\omega_{i,j}^{\alpha\beta} := \delta_i^\alpha \delta_j^\beta - \delta_j^\alpha \delta_i^\beta \in \Omega,$$

where δ denotes the Kronecker symbol

$$\delta_\alpha^i = \begin{cases} 1 & \text{if } \alpha = i, \\ 0 & \text{else.} \end{cases}$$

The claim (B.1) then follows, if we can show that there exists a uniform constant $c > 0$ such that for any $p \equiv (p_i)_{i=1}^N$, $q \equiv (q_i)_{i=1}^N \in \mathbb{R}^N$ with $|p| = |q| = 1$, and $p \perp q \in \mathbb{R}^N$, there are $\alpha \neq \beta$ such that

$$|\langle p_{\omega^{\alpha\beta}}, q \rangle_{\mathbb{R}^N}| \equiv |p_\alpha q_\beta - p_\beta q_\alpha| \geq c.$$

Choose $i_0 \neq k_0$ such that (w.l.o.g) $p_{i_0}, q_{k_0} > 0$ and, more importantly,

$$p_{i_0}, q_{k_0} \geq \frac{1}{2\sqrt{N-1}} =: c_0. \quad (\text{B.2})$$

In fact, by orthonormality of $p \perp q$, it cannot happen, that for some i , both, $|p_i|^2, |q_i|^2 \geq \frac{3}{4}$. Thus, if there is some i_0 such that $|p_{i_0}|^2 \geq \frac{3}{4}$, then $q_{i_0}^2 \leq \frac{3}{4}$ and because of $|q| = 1$ there has to exist at least one $k_0 \neq i_0$ such that q_{k_0} satisfies (B.2). An analogous argument holds, if $|q_{k_0}|^2 \geq \frac{3}{4}$ for some k_0 . In the remaining cases, we can assume that $|p_i|^2, |q_k|^2 < \frac{3}{4}$ for all i, k . But then there have to be at least two indices $i_0 \neq i_1$ such that p_i satisfies (B.2) for $i = i_0, i_1$: because if this was not the case, we had the following contradiction:

$$1 = \sum_{i=1}^N |p_i|^2 < \frac{3}{4} + (N-1) \frac{1}{4(N-1)} = 1.$$

We thus have two different choices for both for i_0 in order for p_{i_0} to satisfy (B.2), so whichever the choice of k_0 is, we can choose $i_0 \neq k_0$. Then,

$$\langle p_{\omega^{i_0 k_0}}, q \rangle_{\mathbb{R}^N} = p_{i_0} q_{k_0} - p_{k_0} q_{i_0}.$$

If $p_{k_0} q_{i_0} \leq 0$, this implies,

$$\langle p_{\omega^{i_0 k_0}}, q \rangle_{\mathbb{R}^N} \geq (c_0)^2.$$

Assume on the other hand, that $p_{k_0}q_{i_0} > 0$ and that even

$$\langle p_{\omega^{i_0 k_0}}, q \rangle \leq \frac{1}{10}(c_0)^2.$$

Then,

$$|p_{k_0}q_{i_0}| = p_{k_0}q_{i_0} = p_{i_0}q_{k_0} - \langle p_{\omega^{i_0 k_0}}, q \rangle_{\mathbb{R}^N} \geq (c_0)^2 - \frac{1}{10}(c_0)^2.$$

Since $|p_{k_0}|, |q_{i_0}| \leq 1$, we infer

$$|p_{k_0}|, |q_{i_0}| \geq \frac{9}{10}(c_0)^2.$$

It follows, that

$$c_1 := \frac{9}{10}(c_0)^3 \leq |q_{i_0}||p_{i_0}| + |q_{k_0}||p_{k_0}|. \quad (\text{B.3})$$

Because of orthogonality $p \perp q$,

$$-(q_{i_0}p_{i_0} + q_{k_0}p_{k_0}) = \sum_{i \neq i_0, k_0} p_i q_i. \quad (\text{B.4})$$

As the product $p_{k_0}q_{i_0} > 0$ we have to consider the case *I*, where both, $p_{k_0}, q_{i_0} > 0$ and the case *II* where both, $p_{k_0}, q_{i_0} < 0$. Recall that in both cases $p_{i_0}, q_{k_0} > 0$. As for case *I*, (B.4), (B.3) imply

$$-c_1 \geq \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i < 0}} p_i q_i + \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i > 0}} p_i q_i = - \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i < 0}} |p_i||q_i| + \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i > 0}} |p_i||q_i| \geq - \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i < 0}} |p_i||q_i|.$$

In particular, this implies, that there is $i_1 \neq i_0, k_0$ such that $p_{i_1}q_{i_1} < 0$ and still (recall $|p_{i_1}|, |q_{i_1}| \leq 1$)

$$|p_{i_1}|, |q_{i_1}| \geq \frac{c_1}{N}.$$

Assuming $p_{i_1} > 0, q_{i_1} < 0$, since we know $q_{i_0} > 0, p_{i_0} > 0$

$$\langle p_{\omega^{i_1 i_0}}, q \rangle_{\mathbb{R}^N} = p_{i_1}q_{i_0} - p_{i_0}q_{i_1} > |p_{i_0}||q_{i_1}| \geq c_0 \frac{c_1}{N}.$$

If we assume, on the other hand, $p_{i_1} > 0, q_{i_1} < 0$,

$$-\langle p_{\omega^{i_1 i_0}}, q \rangle_{\mathbb{R}^N} = -p_{i_1}q_{i_0} + p_{i_0}q_{i_1} < -|p_{i_0}||q_{i_1}| \leq -c_0 \frac{c_1}{N}.$$

In the case *II*, where both $p_{k_0}, q_{i_0} < 0$, we have instead

$$c_1 \leq - \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i < 0}} |p_i||q_i| + \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i > 0}} |p_i||q_i| \leq \sum_{\substack{i \neq i_0, k_0 \\ p_i q_i > 0}} |p_i||q_i|,$$

and we can find $i_2 \neq i_0, k_0$ such that $p_{i_2}q_{i_2} > 0$ and

$$|p_{i_2}|, |q_{i_2}| \geq \frac{c_1}{N}.$$

Then,

$$-\text{sign}(p_{i_2}) \langle p_{\omega^{i_2 i_0}}, q \rangle_{\mathbb{R}^N} = |p_{i_2}||q_{i_0}| + p_{i_0}|q_{i_2}| \geq c_0 \frac{c_1}{N}.$$

This proves the first claim, which essentially is contained in the following: For $|p| = 1$, the finite set $\{p_\omega, \omega \in \Omega\}$ may not be a basis; Nevertheless, it is a linear generator of the space $p^\perp \subset \mathbb{R}^N$.

For the second claim we use the following argument: Since Ω is finite, any vector q of length $|q| = 1$ has to have length at least $\theta = \frac{1}{|\Omega|+1}$ in at least one of the linear spaces generated by some p_ω or generated by p itself. That is $|\langle p_\omega, q \rangle| \geq \theta$ for some $\omega \in \Omega$, or $|\langle p, q \rangle| \geq \theta$. Consequently, if there is some ω such that $|\langle p_\omega, q \rangle| < \theta$, then there has to be another $\tilde{\omega} \in \Omega$ such that $|\langle p_{\tilde{\omega}}, q \rangle| \geq \theta$ or, alternatively, $|\langle p, q \rangle| \geq \theta$. \square

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